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WADD TECHNICAL REPORT

SPATIAL FILTERING

by

Frank Ayres

Antenna Laboratory
Department of Electrical Engineering
The Ohio State University

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ABSTRACT

The mathematical relationships for filtering in a space domain are developed in a general analogy with temporal filtering. Cases in which the elements of the filtering system are continuous or discrete are discussed. General examples of spatial filtering and the realization of such filters are shown.

TABLE OF CONTENTS

FOREWORD

This report was prepared by The Ohio State University Antenna Laboratory, on Air Force Contract AF 33(616)-7843, under Task Number 50682 of Project Number 1(620-4144), "Guidance and Sensing Techniques for Advanced Vehicles". The work was administered under the direction of Navigation and Guidance Laboratory, Aeronautical Systems Division, Air Force Systems Command, United States Air Force, Wright-Patterson Air Force Base, Ohio. The task engineer for the laboratory was

CHAPTER I
CONTINUOUS FUNCTIONS

A. Introduction

Spatial filtering is a method by which a particular object or set of objects may be highlighted so that detection is improved with a minimum loss of information. When a signal is applied to a device which changes the characteristics of this input signal, a filtering process has been performed. Spatial filtering thus implies that a field is processed to create a new field. The desired relationship between the new and the original fields specifies the nature of the filtering to be accomplished.

There is a close analogy between linear spatial filtering and linear temporal filtering; however, many of the restrictions of temporal filtering do not apply to spatial filtering, such as:

1. The output signal in a time filtering circuit cannot appear until the input has appeared. No such limitation applies to spatial filtering.

2. Spatial filtering techniques may be applied to any number of dimensions.

It is thus apparent that spatial filtering is inherently more flexible than time filtering.
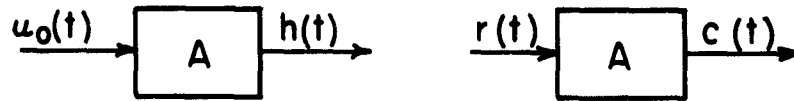
B. Impulse Response

Fig. 1.   Temporal filter.

Figure 1 represents a temporal filter in which time is the single independent variable. $u_o(t)$ represents a unit impulse and $h(t)$ is the characteristic time response to a unit impulse. If the filter A is linear and stable, superposition principles may be applied and the relationship between an arbitrary input or reference signal $r(t)$ and the filtered output signal $c(t)$ can be characterized by a superposition integral:

$$(1) \qquad c(t) = \int_{-\infty}^{+\infty} r(\tau)\, h(t-\tau)\, d\tau$$

It can be shown that the Laplace transform of $r(t)$ and $c(t)$, designated as $R(S)$ and $C(S)$ respectively, are related by an equation of the form:

$$(2) \qquad C(S) = H(S)\, R(S)$$

$H(S)$ is normally referred to as the transfer function of the filter and is the Laplace transform of $h(t)$. Thus by means of the convolution integral in Equation (1), the output can be determined in real time by using time functions rather than frequency functions.

In a more general view, filtering may be considered for a source signal with more than one independent variable, as for example, x and y space coordinates. Such a filtering device may be a lens or similar optical device in which the spatial features of the object, such as the radiance, determine the output image. As stated previously, spatial filtering techniques may be applied to three or more dimensions; however, the concepts developed here will be limited to two dimensional signals as a matter of convenience.

Let r and c be considered now as functions of x and y , where r is again the reference or input signal and c is the output or resultant image. The convolution or superposition integral for a filter which is linear and independent of x and y is given by:

$$(3) \qquad c(x,y) = \int_{x'=-\infty}^{+\infty} \int_{y'=-\infty}^{+\infty} r(x', y') h(x - x', y - y') dx' dy'$$

h(x, y) is the spatial impulse response of the filter. Note that the integrals range from $-\infty$ to $+\infty$.

C. Frequency Response

The frequency response of continuous functions may be determined by means of either the Fourier transform or the Laplace transform. It is usually more advantageous to use the Fourier transform in a general case because the range of the integrals includes the negative

3

region as well as the positive region. The Fourier transform for the temporal case is defined as:

$$(4) \qquad F(j\omega) = \int_{-\infty}^{+\infty} f(t) \, e^{-j\omega t} dt$$

The Fourier transform of $r(x, y)$, $c(x, y)$, and the impulse response $h(x, y)$ can be written:

$$(5) \qquad C(j\omega_x, j\omega_y) = R(j\omega_x, j\omega_y) H(j\omega_x, j\omega_y)$$

Here ω_x and ω_y are termed spatial frequencies. The transforms are determined from Equation (4), as for example:

$$(6) \qquad H(j\omega_x, j\omega_y) = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{+\infty} h(x, y) \, e^{-j(\omega_x x + \omega_y y)} \, dx \, dy$$

Because of the two-sided nature of the transforms, initial conditions are not considered.

The inverse transform of $C(j\omega_x, j\omega_y)$ is given by:

$$(7) \qquad c(x, y) = \frac{1}{(2\pi)^2} \int \int C(j\omega_x, j\omega_y) \, e^{+j(\omega_x x + \omega_y y)} \, d\omega_x \, d\omega_y$$

The problems of determining either the direct or inverse transforms are greatly simplified if the variable r and h can be written in product form

$$(8) \qquad r(x, y) = r_x(x) \, r_y(y)$$

4

(9) $$h(x,y) = h_x(x) \, h_y(y)$$

or as sums of products

(10) $$r(x,y) = \sum_n \sum_m r_{nx}(x) \, r_{my}(y)$$

(11) $$h(x,y) = \sum_n \sum_m h_{nx}(x) \, h_{my}(y)$$

The transform of Equation (9) for example is then given by

(12) $$H(j\omega_x, \, j\omega_y) = \left[\int_{-\infty}^{+\infty} h_x(x) \, e^{-j\omega_x x} \, dx \right] \left[\int_{-\infty}^{+\infty} h_y(y) \, e^{-j\omega_y y} \, dy \right]$$

and the transform of c becomes simply

(13) $$C(j\omega_x, \, j\omega_y) = F_1 \, (j\omega_x) \, F_2 \, (j\omega_y)$$

where F_1 is the product of the transforms of $r_x(x)$ and $h_x(x)$ and F_2 is the product of the transforms of $r_y(y)$ and $h_y(y)$. Unfortunately, separation of the functions is not always possible. Transforms such as Equation (12) can be found in the conventional transform tables and are generally double-sided transforms.

D. Spot Size

A spot can be considered as a continuous function of x and y. An electron beam striking the face of a cathode ray tube is an example of point sources acting on a medium and the response can be described in x, y coordinates as :

(14) $$h(x,y) = e^{-k(x^2 + y^2)}$$

5

In order to determine the spatial frequency response of the tube, $H(j\omega_x, j\omega_y)$ must be determined.

$$(15) \quad H(j\omega_x, j\omega_y) = \int\int_{-\infty}^{+\infty} e^{-k(x^2 + y^2)} \, e^{-j(\omega_x x + \omega_y y)} \, dx \, dy$$

This can be separated as

$$(16) \quad H(j\omega_x, j\omega_y) = \int_{-\infty}^{+\infty} e^{-ky^2} \left[\int_{-\infty}^{+\infty} e^{-(kx^2 + j\omega_x x)} \, dx \right] e^{-j\omega_y y} dy$$

Now, it is known from the Gaussian distribution function that the following relationship exists:

$$(17) \quad \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}} \, dx = 1$$

Equation (17) can be used to evaluate Equation (16) if we let $2\sigma^2 = 1$. Then $\sigma = \sqrt{1/2}$

and

$$(18) \quad \frac{1}{\sqrt{2\pi}\left(\frac{1}{\sqrt{2}}\right)} \int_{-\infty}^{+\infty} e^{-x^2} dx = 1$$

or

$$(19) \quad \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Consider the bracket term $\left[\int_{-\infty}^{+\infty} e^{-(kx^2 + j\omega_x x)} dx \right]$ in (16). Let $x = a + z$, then $dx = dz$ and

$$kx^2 + j\omega_x x = ka^2 + 2azk + kz^2 + j\omega_x a + j\omega_x z$$

6

Let $2akz + j\omega_x z = 0$, or $a = -j\dfrac{\omega_x}{2k}$

Then $kx^2 + j\omega_x x = kz^2 + \dfrac{\omega_x^2}{4k}$

The bracket term then becomes

(20)
$$\int_{-\infty}^{+\infty} e^{-(kz^2 + \frac{\omega_x^2}{4k})} dz = e^{-\frac{\omega_x^2}{4k}} \int_{-\infty}^{+\infty} e^{-kz^2} dz$$

Let

$k = r^2$ and $kz^2 = r^2 z^2 = m^2$, then $dz = \dfrac{dm}{r}$

(21)
$$\frac{e^{-\frac{\omega_x^2}{4r^2}}}{r} \int_{-\infty}^{+\infty} e^{-m^2} dm = \frac{\sqrt{\pi} \, e^{-\frac{\omega_x^2}{4r^2}}}{r}$$

or

(22)
$$\frac{\sqrt{\pi} \, e^{-\frac{\omega_x^2}{4k}}}{\sqrt{k}}$$

The same operation performed on the y terms yields

(23)
$$\int_{-\infty}^{+\infty} e^{-(ky^2 + j\omega_y y)} dy = \frac{\sqrt{\pi} \, e^{-\frac{\omega_y^2}{4k}}}{\sqrt{k}}$$

Thus

(24)
$$H(j\omega_x, j\omega_y) = \frac{\pi \, e^{-\frac{(\omega_x^2 + \omega_y^2)}{4k}}}{k}$$

7

CHAPTER II
DISCRETE CASE

In any filtering process, the characteristic response of the filter to a unit impulse is used to determine the output for an arbitrary input signal. In a discrete filter, this characteristic response is defined as a matrix $A = \left[a_{pq} \right]$. Since p and q can have both positive and negative values, conventional notation cannot be used in which the upper left matrix element is designated as a_{11} . It is therefore necessary to indicate the values of p and q for at least one position in the matrix. In the convention used here, each matrix position is divided diagonally, with the position notation in the upper left segment and the element value in the lower right segment. Thus a typical A matrix could be that shown in Figure 2.

Fig. 2. $\underline{A}$ Matrix for spatial filtering.

In Section I, the case of continuous functions was treated wherein the object, filter, and resulting image were each continuous in x and y . The spatial filtering techniques described in this section will make use of the discrete A matrix as a filter; however the object and image may be either discrete or continuous.

8

Before discussing the filtering process itself, it is worthy to note some of the characteristics of this A matrix.

1. If the $\sum_p \sum_q a_{pq} = 0$, there is a zero average value for the output when this matrix is used as a filter. In other words, the dc level is not passed in such a filter.

2. If the $\sum_p \sum_q a_{pq} \neq 0$, the dc level is passed and the image will have an average value taken over all cells which is not equal to zero.

3. Certain configurations of values in the A matrix permit expressing the matrix as $A = MN$, where M indicates a column matrix and N indicates a row matrix. (The proof of this is in Section II B.) If this is possible, the terms can be treated separately as functions of x only and functions of y only. As a matter of convenience, the discussions which follow will be restricted to this type of matrix.

An example of 3 above is shown in Figure 3.

$$\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

$$\sum_{p,q} A_{pq} = 0$$

(a) **A Matrix As A Column And Row Matrix**   (b) **Resultant A Matrix**
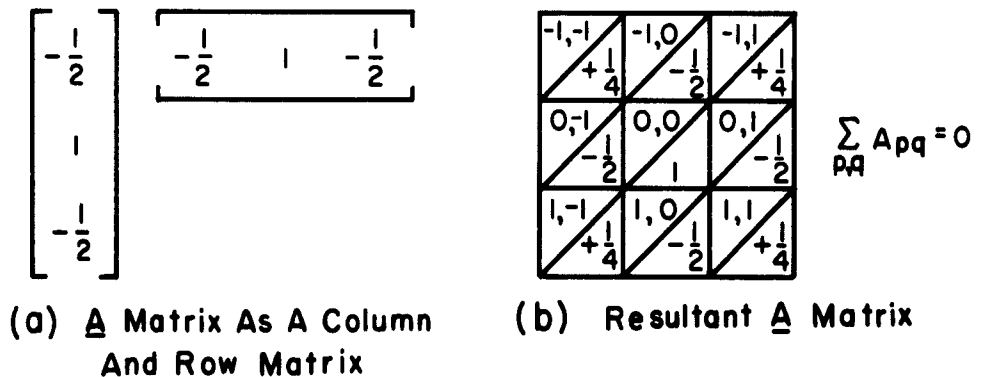
Fig. 3.   Factoring an A Matrix.

9

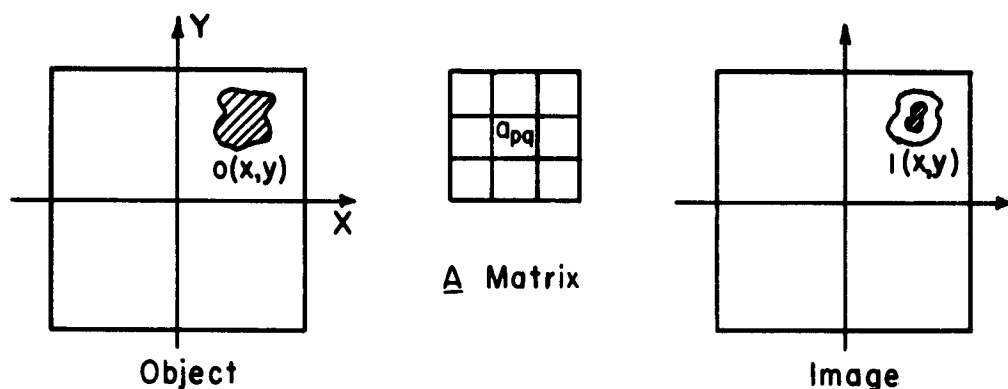## A. Discrete A Matrix, Continuous Object and Image

Fig. 4. Filtering a continuous object.

An individual scene can be described completely as far as the spatial distribution of radiance is concerned. The two dimensional function $R(x, y)$, where R is a radiance, gives all the available information. The information is usually collected by some type of sensor which is limited in the resolution of surface detail. This characteristic of the sensor will be defined here as the cell size of the sensor. The cell dimensions will be defined as the minimum size of cell such that two identical objects, each located in a similar position in adjacent cells, will always be separated or resolved by the sensor. In the case of a ground-based radar, the cell size is given by $c\tau/2$ in range and $r \Delta \phi$ in azimuth, where c is the velocity of light, τ is the pulse width, r is the range, and $\Delta \phi$ is the beamwidth of the antenna in azimuth. In the case of the image orthicon or similar optical transducers, the cell size must be defined in terms of the size of the electron

10

beam used to scan the image projected upon the sensitized surface within the pickup tube.

Figure 4 depicts a continuous object which is to be filtered by the discrete A matrix. This continuous object, $O(x,y)$, may be pictured as an aerial view of terrain in which contours are functions of x and y; or intensity levels in an infrared photograph would yield continuous functions of x and y. The resultant image is labeled $I(x,y)$.

In order to perform spatial filtering on this continuous object, $O(x,y)$, two approaches may be considered.

1. The object may be sampled to yield discrete values for the x, y positions. This quantized object may then be operated upon by the A matrix using a convolution process to produce a discrete image $I(x,y)$. This convolution process is discussed in Section II B. By use of a $\frac{\sin x}{x}$ type filter or a holding circuit, the discrete image can be changed to a continuous function. It might be pointed out that the human eye can also perform this discrete-to-continuous operation as evidenced by a photograph in a newspaper, which is made up of a large number of closely spaced dots, appearing to be of a continuous nature when viewed from reading distance.

2. The A matrix may be transformed into a continuous function by describing it as a series of step functions. The filtering process is then the same as in Section I where the object and filter were

11

continuous. As an example of this process, consider the A matrix in Figure 3. This can be represented as :
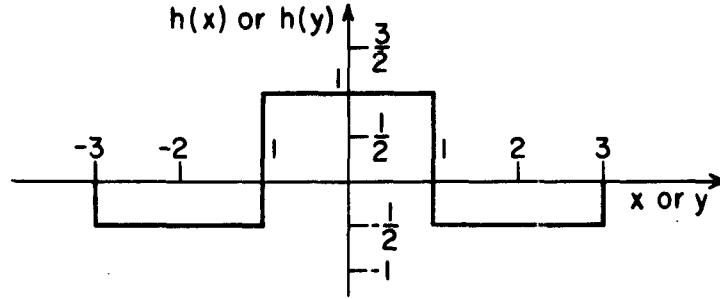

Fig. 5. The A Matrix described as step functions.

The equations may be written for Figure 5 as:

$$(25) \quad H_x(j\omega_x) = \frac{(e^{j\omega_x} - e^{-j\omega_x}) - \frac{1}{2}(e^{j3\omega_x} - e^{j\omega_x}) - \frac{1}{2}(e^{-j\omega_x} - e^{-j3\omega_x})}{j\omega_x}$$

Collecting terms

$$(26) \quad H_x(j\omega_x) = \frac{3}{2}\left[\frac{e^{j\omega_x} - e^{-j\omega_x}}{j\omega_x}\right] - \frac{1}{2}\left[\frac{e^{j3\omega_x} - e^{-j3\omega_x}}{j\omega_x}\right]$$

$$(27) \quad H_x(j\omega_x) = 3\left[\frac{\sin \omega_x}{\omega_x} - \frac{\sin 3\omega_x}{3\omega_x}\right]$$

$H_y(j\omega_y)$ is obtained here by substituting ω_y in Equation (27) for ω_x since the matrix is symmetrical about the center cell. This expression illustrates how the x and y spatial frequencies are filtered. It corresponds roughly to a bandpass filter except that it introduces no phase shift. The function is sketched in Figure 6.
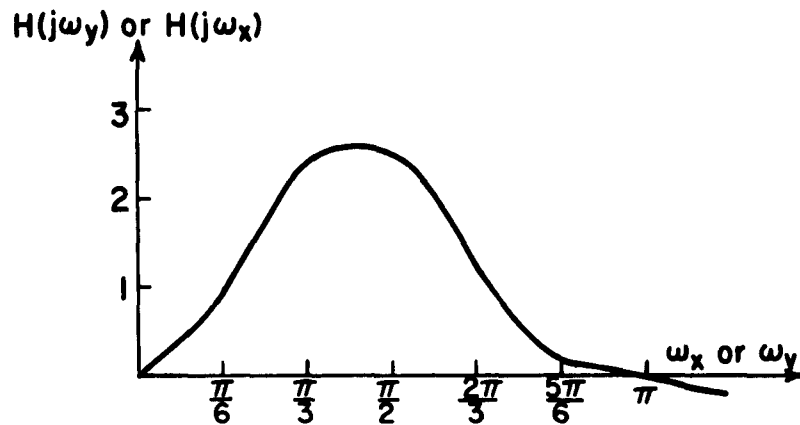
12

H(jω_y) or H(jω_x)

Fig. 6. Frequency response of the A Matrix
as a function of ω_x or ω_y.

To determine $I(x, y)$ when using this filter, the Fourier trans-

form of the object, $O(j\omega_x, j\omega_y)$, is determined as described in Section

I.

$$(28) \quad O(j\omega_x, j\omega_y) = \int_{-\infty}^{+\infty} \int O(x, y) \, e^{-j(\omega_x x + \omega_y y)} dx \, dy$$

$$(29) \quad I(j\omega_x, j\omega_y) = H(j\omega_x, j\omega_y) \, O(j\omega_x, j\omega_y)$$

$$(30) \quad I(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int I(j\omega_x, j\omega_y) \, e^{j(\omega_x x + \omega_y y)} d\omega_x \, d\omega_y$$

Before method 1 above may be employed, the continuous object

must first be sampled. It must be kept in mind that the sampling

frequency must be at least twice the highest frequency of the input

13

signal in order to recover the full information content with a linear

device which can separate only on the basis of differences in the

frequency spectrum. In a spatial object, the variations in the x,y

plane correspond to amplitude variations in a time signal, which is

one dimensional.

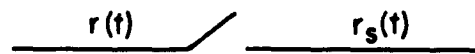$$\underline{\qquad \dfrac{r(t)}{\qquad}\diagup\dfrac{r_s(t)}{\qquad}}$$

Fig. 7. A simple sampler.

Consider a simple time sampler as shown in Figure 7.

Figure 8 depicts sampling as modulation of a pulse train where the

amplitude of the sampler output is derived from the value of the input

signal at any given time. An impulse occurring at t = nT would have

a Laplace transform of

(31) $L[u_0(t - nT)] = e^{-nTS}$.

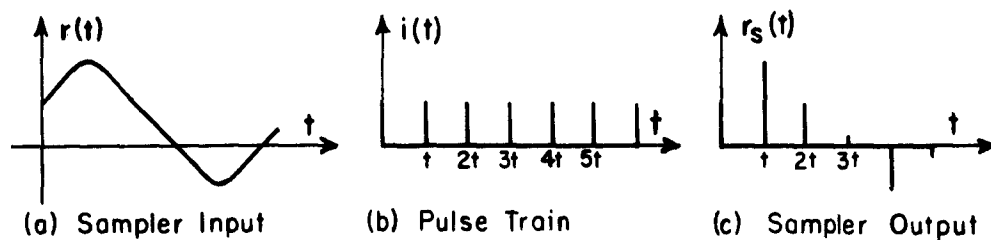(a) Sampler Input   (b) Pulse Train   (c) Sampler Output

Fig. 8. Sampling considered as modulation
of pulse train.

14

This is seen to be a shifting function, or delay. Thus the sampler output at $t = nT$ would be $r(nT)e^{-nTS}$. The shifting function form is used in the Z transform by defining

$$(32) \qquad\qquad Z = e^{ST}$$

To sample the spatial object, the same sampling method may be used except that two dimensional shifting is required. In this case, the shifting function Z is defined as

$$(33) \qquad Z_x = e^{S_x \Delta X} \qquad \text{and} \qquad Z_y = e^{S_y \Delta Y}$$

where ΔX and ΔY are determined by the sensor resolution.

Now the image, $I_{m,n}$, which results from a sampled object, $O_{m,n}$, and a discrete filter can be expressed as

$$(34) \qquad I_{m,n} = \sum_k \sum_p O_{m-k,n-p}\, a_k\, a_p$$

where a_k and a_p may be considered as weighting functions derived from the impulse response of the A matrix filter. Only symmetrical discrete filters are considered in the derivations here and the weighting functions can therefore be treated independently. A spatial filter will usually be symmetrical to avoid undesired distortion of the object. It must be remembered that a filter is designed to produce a desired output from a given input.

15

Z transforms may be used to describe the shifting and the transform of the object becomes

$$(35) \qquad G_o(Z_x, Z_y) = \sum_m \sum_n O_{m,n} Z_x^{-m} Z_y^{-n}$$

The transform of the image is

$$(36) \qquad G_I(Z_x, Z_y) = \sum_m \sum_n I_{m,n} Z_x^{-m} Z_y^{-n}$$

The weighting functions of the filter become

$$(37) \qquad A_x(Z_x) = \sum_m a_m Z_x^{-m}$$

$$(38) \qquad A_y(Z_y) = \sum_n a_n Z_y^{-n}$$

Then

$$(39) \qquad G_I(Z_x, Z_y) = A_x(Z_x) A_y(Z_y) G_o(Z_x, Z_y)$$

As an example of this type of transform, consider the A Matrix used previously in Figure 3. The Z transform would be

$$(40) \qquad A_x(Z_x) = -\tfrac{1}{2} Z_x^{-1} + 1 Z_x^0 \cdot -\tfrac{1}{2} Z_x^{+1}$$

$A_y(Z_y)$ would be the same except for subscripts.

$$(41) \qquad A_x(Z_x) = -\tfrac{1}{2} \left[\frac{(Z_x - 1)^2}{Z_x} \right]$$

16

$$(42) \qquad A_y(Z_y) = -\tfrac{1}{2}\left[\frac{(Z_y-1)^2}{Z_y}\right]$$

$$(43) \qquad A_x(Z_x)A_y(Z_y) = \tfrac{1}{4}\left[\frac{(Z_x-1)^2}{Z_x}\frac{(Z_y-1)^2}{Z_y}\right]$$

17

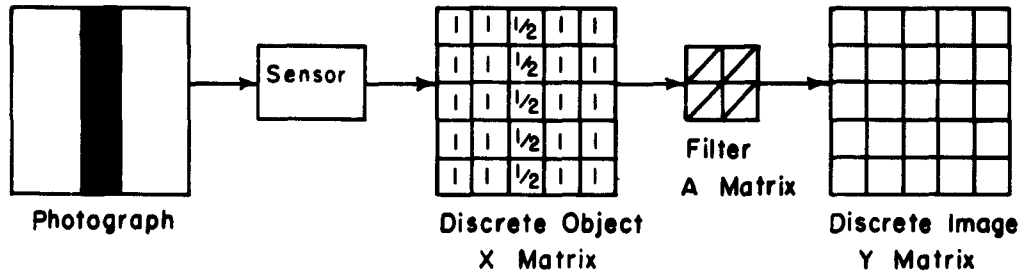## B. Discrete A Matrix, Discrete Object and Image

Fig. 9. Filtering a discrete object.

Consider a system such as that shown in Figure 9 where an object, such as a photograph, is scanned by a sensor. The output of the sensor is sampled as described in the previous section to yield discrete values for each cell in the storage location, which shall be called the X Matrix. Each position of this X Matrix then corresponds to an x, y position in the original photograph and the value of the cell corresponds to the amplitude of the photograph (for example, the intensity) at that position. It can be expected that noise will be introduced along with the signal so that ideal values as shown in the discrete object will not be realized. Filtering by a discrete A Matrix filter will produce a discrete image, which shall be designated the Y Matrix.

The system in Figure 9 could be reduced by performing the desired filtering as the information is being stored in the X Matrix,

but it seems more convenient to explain the filtering action in the system as shown. Thus, the system now consists of a discrete object, filter, and image.

As mentioned before, the characteristics of the spatial filter are based on the desired final product - the image. Functions which can be performed spatially include averaging, differentiation, and integration. Use is made of averaging to remove fine detail of a field but extended structure is retained. Differentiation can be used to retain sharp boundaries while the background is removed.

The Y Matrix will have the same number of rows and columns as the X Matrix but it will contain the desired features of the X Matrix with the redundancies and undesired information removed. The information stored in cell y_{ij} of the Y Matrix may be expressed by

$$(44) \qquad y_{ij} = \sum_p \sum_q a_{pq} \, x_{i+p, \, j+q}$$

It can be seen that this requires a shifting process. For this purpose a shifting matrix may be used by defining

$$S_g = s_{ij} \, , \quad s_{ij} = \begin{cases} 1, & i = j - g \\ 0, & \text{otherwise} \end{cases}$$

where g is a constant integer which can be either positive or negative. The shifting direction is determined by the sign of g and whether S_g

occurs before or after the matrix to be shifted. As an example:

$$B = S_g \, C$$

or

$$b_{ik} = \sum_j s_{ij} \, c_{jk}$$

By the definition of s_{ij}, the only term which has a value is the one for which $j = i + g$; thus

$$b_{ik} = c_{i + g, k}$$

When

$$B = CS_g \, , \quad b_{kj} = \sum_i c_{ki} \, s_{ij} = c_{k, j - g}$$

With g a <u>positive</u> integer, it can be seen that when the shifting matrix appears first, the resultant matrix is identical to the original matrix except that all elements have been shifted <u>up</u> g rows with g rows of zeros at the bottom of the resultant matrix. When S_g appears after the original matrix, the resultant matrix elements have all been shifted g columns to the <u>right</u> with g columns of zeros on the left. S_o is the identity matrix.

To demonstrate this shifting operation, take the case for <u>negative</u> shifting by letting $g = -1$.

$$B = S_{-1} \, C \qquad b_{ik} = c_{i-1, k}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Note that C has been shifted <u>down</u> one row to form B.

$$B = CS_{-1} \qquad b_{kj} = c_{k,\,j+1}$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 0 \\ 6 & 7 & 8 & 0 \\ 10 & 11 & 12 & 0 \\ 14 & 15 & 16 & 0 \end{bmatrix}$$

Note that C has shifted <u>left</u> one column to form B.

Thus, column and row shifting of the X Matrix can be performed by the process

$$S_p \, X S_q$$

which shifts the original X Matrix up p rows and q columns right (p and q positive integers). Equation (44) can therefore be written in matrix form as

$$(45) \qquad Y = \sum_p \sum_q a_{pq} \, S_p \, X S_{-q}$$

As stated previously, and shown in Figure 3, certain arrangements of the A matrix elements lend themselves to column and row factoring. If such is the case, so that $A = MN$, where $M = m_{p,1}$ is a column matrix and $N = n_{1,q}$ is a row matrix, Equation (45) can be expressed in the form

21

$$(46) \qquad Y = \sum_{p} \sum_{q} m_p S_p \, X \, S_{-q} n_q$$

since $a_{pq} = m_p n_q$. The summations are now independent of the matrix multiplications so

$$Y = \sum_{p} m_p S_p \, X \sum_{q} S_{-q} n_q$$

or

$$(47) \qquad\qquad Y = A^{\circ} \, X \, A'$$

when

$$A^{\circ} = \sum_{p} m_p S_p \qquad \text{and} \qquad A' = \sum_{q} S_{-q} n_q$$

It is usually desirable to find a single A matrix which performs the same function as the sequential application of A° and A' . This can be symbolized by

$$(48) \qquad\qquad A = A^{\circ} * A'$$

The * operation must satisfy the operations for an $X = x_{pq}$ matrix such that

$$x_{pq} = \begin{cases} 1, & p = a; \ q = b \\ 0, & \text{otherwise} \end{cases}$$

When X is operated on with A only, the following result is obtained:

$$y_{a-i, b-j} = \sum_{p} \sum_{q} a_{pq} x_{a-i+p, b-j+q} = a_{ij}$$

22

An operation on X with A^O and A' yields

$$y_{a-i,b-j} = \sum_u \sum_v a^O_{u,v} \sum_c \sum_d a'_{c,d} \; x_{a-i+u+c,b-j+v+d}$$

$$= \sum_u \sum_v a^O_{u,v} \; a'_{i-u,j-v}$$

Therefore

$$(49) \qquad a_{ij} = \sum_u \sum_v a^O_{u,v} a'_{i-u,\,j-v}$$

It is thus seen that for $A = A^O * A'$, the $*$ operation is a two dimensional convolution operation.

If A^O and A' are both factorable into one row and one column matrices, so that $A^O = M^O N^O$ and $A' = M'N'$, then the convolution of A^O and A' may be expressed as

$$a_{ij} = \sum_u \sum_v m^O_u n^O_v m'_{i-u} n'_{j-v}$$

Since the summations are independent

$$a_{ij} = \sum_u m^O_u m'_{i-u} \sum_v n^O_n n'_{j-v} = m_i n_j$$

where

$$m_i = \sum_u m^O_u \, m'_{i-u} \qquad \text{and} \qquad n_j = \sum_v n^O_v \, n'_{j-v}$$

The resultant A matrix can therefore be expressed as $A = MN$. The

23

A matrix is thus factorable when A^0 and A' are factorable and the elements a_{ij} of the A matrix can each be obtained by a one-dimensional convolution.

Desired information can be stored in the Y matrix of Figure 9, while undesired information is rejected, by selecting a specific A matrix to perform the required filtering. It was shown in Figure 6 that filtering comparable to a bandpass temporal filter could be obtained from an A matrix such as that shown in Fig. 3. A low pass filter can likewise be devised which will perform a smoothing (or averaging) function. Consider a normalized A matrix of the form

$$
\begin{bmatrix} \dfrac{1}{3} \\[6pt] \dfrac{1}{3} \\[6pt] \dfrac{1}{3} \end{bmatrix}
\qquad
\begin{bmatrix} \dfrac{1}{3} & \dfrac{1}{3} & \dfrac{1}{3} \end{bmatrix}
$$

$$\sum_{p,q} A_{pq} = 1$$

(a)   Factored Form Of   (b)   A Matrix
      The A Matrix

Fig. 10. A Matrix for spatial smoothing.

The low pass characteristic may be seen from its continuous form as developed in Section II A, pages 11, 12 and 13 .
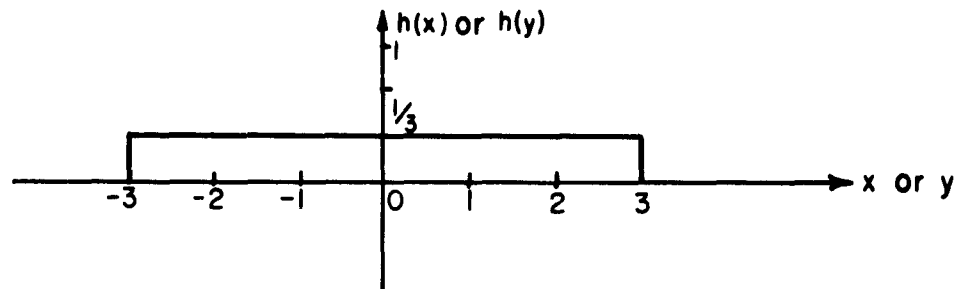
24

Fig. 11. $\underline{A}$ Matrix described as step functions.

$$(50) \qquad H(j\omega_x) = \frac{1}{3} \left[\frac{e^{j3\omega_x} - e^{-j3\omega_x}}{j\omega_x} \right]$$

$$(51) \qquad H(j\omega_x) = 2 \frac{\sin 3\omega_x}{3\omega_x} \quad ; \quad H(j\omega_y) = 2\frac{\sin 3\omega_y}{3\omega_y}$$
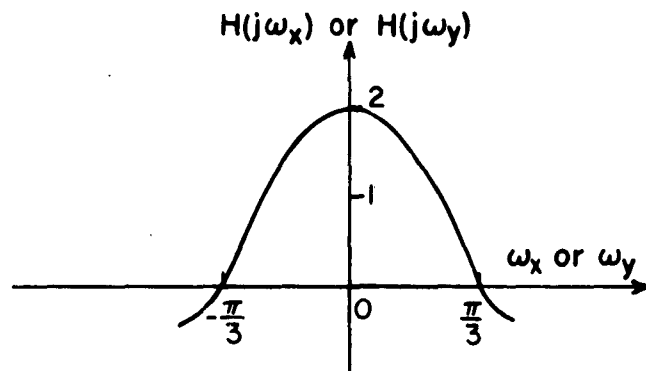

Fig. 12. Frequency response of $\underline{A}$ Matrix as
a function of ω_x or ω_y.

It can thus be seen that an A matrix of the form shown in Figure 10

will exhibit low pass filter characteristics.

Gradient information can also be obtained by the form of the A

matrix. For one-dimensional spatial differentiation, consider the

cells of the A matrix as having a center-to-center spacing of one unit.

To store gradient information in the Y matrix, it is necessary that

(52) $$y_{ij} = \frac{x_{i,j+1} - x_{i,j}}{\Delta u}$$

where Δu is the spacing between columns (and equal to one). If

$\Delta v = 1$ is the spacing between rows, two dimensional gradient inform-

ation may be obtained by

$$y_{i,j} = \frac{1}{2} \left[\frac{(x_{i,j+1} - x_{i,j})}{\Delta u} + \frac{(x_{i+1,j} - x_{i,j})}{\Delta v} \right]$$

(53) $$y_{i,j} = \frac{-x_{i,j} + \frac{1}{2} x_{i,j+1} + \frac{1}{2} x_{i+1,j}}{\Delta u}$$

An A matrix which will perform the operation in Equation (52) would

be as shown in Figure 13 (a). The two-dimensional form in accordance

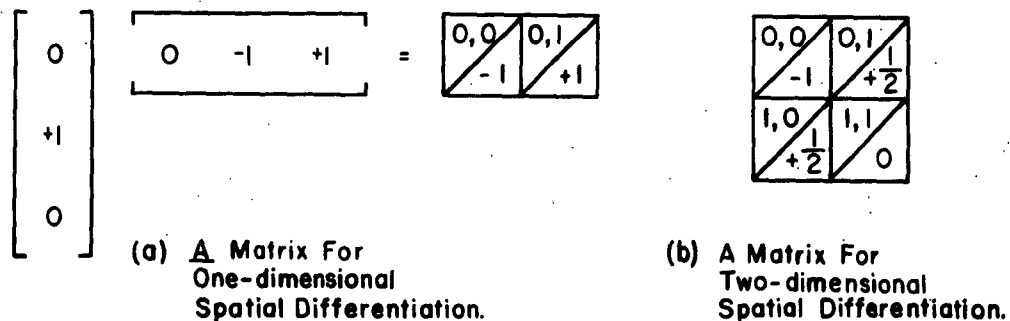with Equation (53) is shown in Figure 13 (b).

(a) **A** **Matrix For**
One-dimensional
Spatial Differentiation.

(b) **A Matrix For**
Two-dimensional
Spatial Differentiation.

Fig. 13. A Matrices for differentiation.

26

The cells values correspond to the coefficients in the equations and the cell positions to the subscripts (x_{ij} is the 0, 0 position).

Consider the advantage of using a one-dimensional gradient type filter where the X matrix has large areas of redundancy. For example:
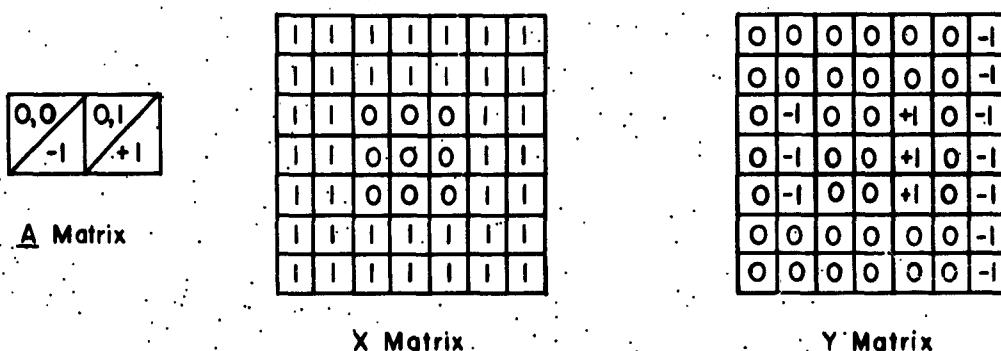

A Matrix   X Matrix   Y Matrix

Fig. 14.   Example of one-dimensional gradient storage.

A row through the centers of the X and Y matrices would appear as shown in Figure 15. The -1 in the far right column of the Y matrix is created by the boundary of the matrix and in a large field it would not be considered. It can be seen that information is stored in the Y matrix only when there is a change in levels between adjacent cells in the X matrix. Hence redundancies are eliminated.

Further examples of spatial filtering with a discrete system are shown in the following section.
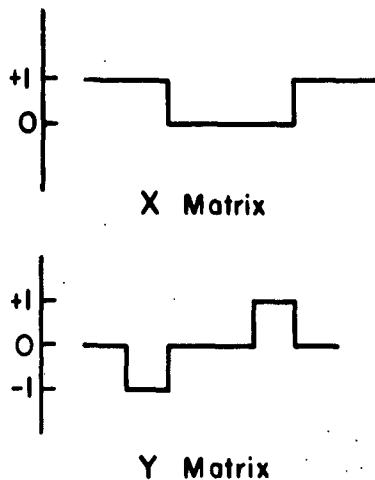
Fig. 15. Result of spatial differentiation.

C. General Examples

The application of the A matrix in Figure 16 to an X matrix, which has a constant background level, is shown for two cases in the following figures. In Figure 17, combinations of single elements which differ from the background are to be filtered. In Figure 21, vertical, horizontal, and diagonal lines are shown against the constant background. The shaded X matrices in Figures 17 and 21 may be considered as "pictures" containing discrete objects or lines against a constant background. These "pictures" are given a relative intensity value for each cell, just as would be done in a sampler (Figures 19 and 23). The filtering action of the A matrix is shown in both "picture"

and "numerical intensity" form in the Y matrices of Figures 18, 20,
22, and 24. In the "pictures", the highest intensity is considered as
white and the lowest intensity as black. Thus, the Y matrix "pictures"
are actually relative intensities, with white and black as the bounds.
The "numerical intensity" figures for the Y matrices have been derived
from the relationship discussed in Section II B and repeated here for
reference

$$y_{ij} = \sum \sum a_{pq} x_{i+p, \; j+q}$$

The periphery values in the Y matrices should be disregarded as they
have little meaning. In the calculation of y_{ij}, the value of zero was
taken for any element outside of the array. For a picture of large
extent, these boundaries would not be so evident.

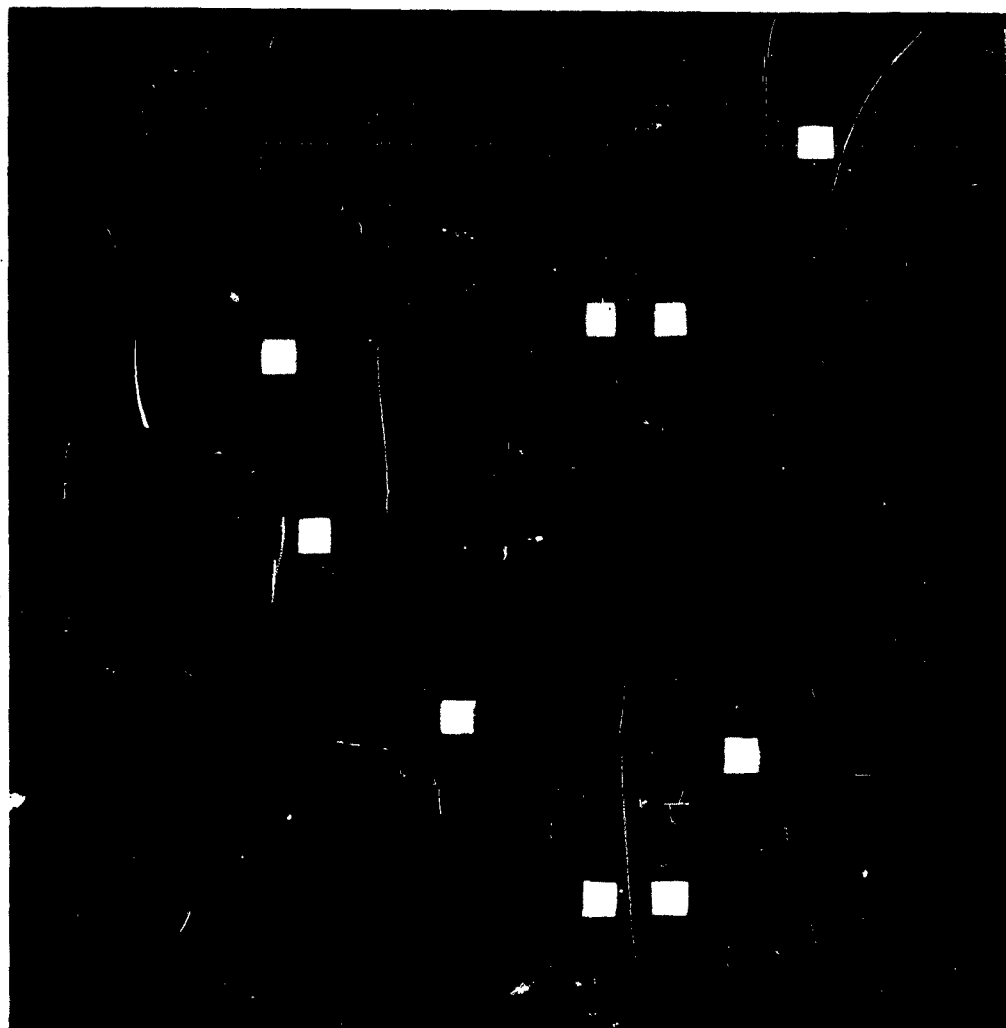Fig. 16. <u>A</u> Matrix for spatial filtering.

Fig. 17. X Matrix.

Fig. 18. Y Matrix which results after filtering Fig. 17.

Fig. 19. Numerical intensity X Matrix for Fig. 17.

32

Fig. 20. Numerical intensity Y Matrix for Fig. 18.

33

Fig. 21. X Matrix.

34
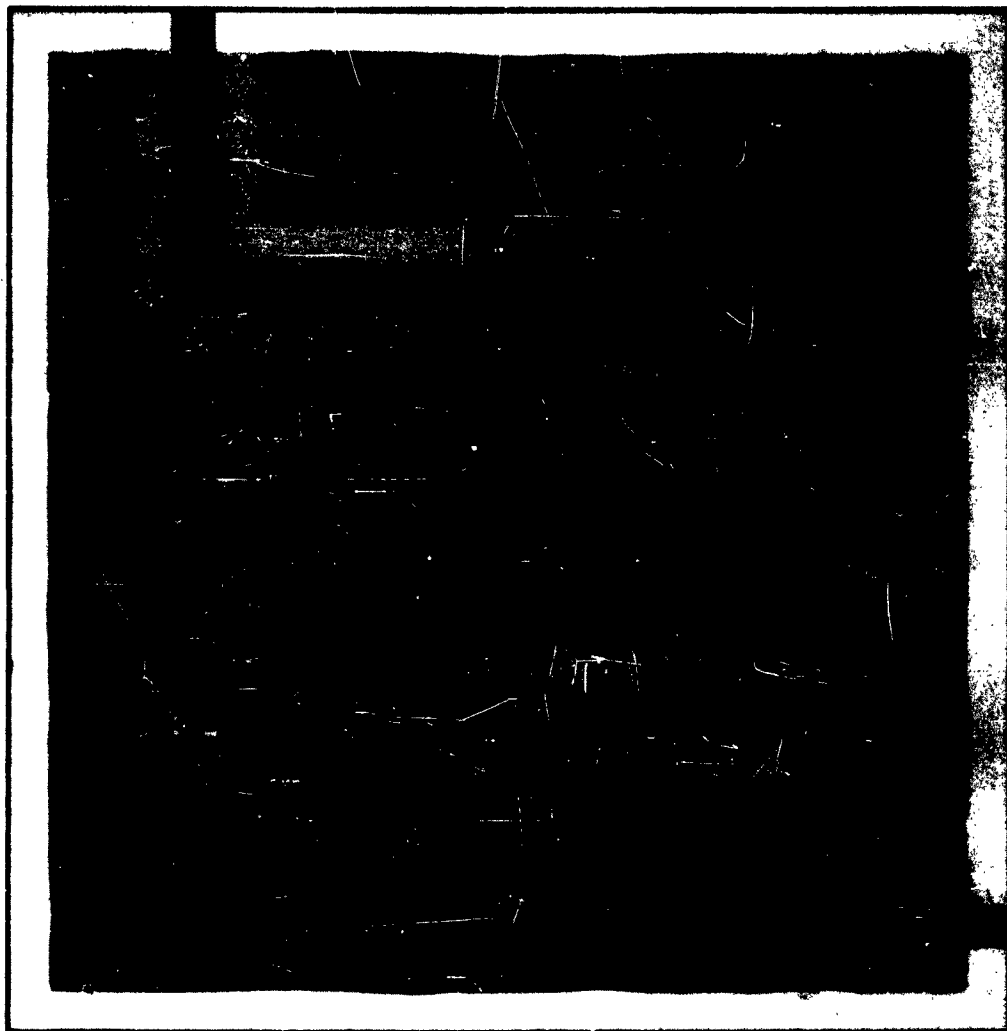
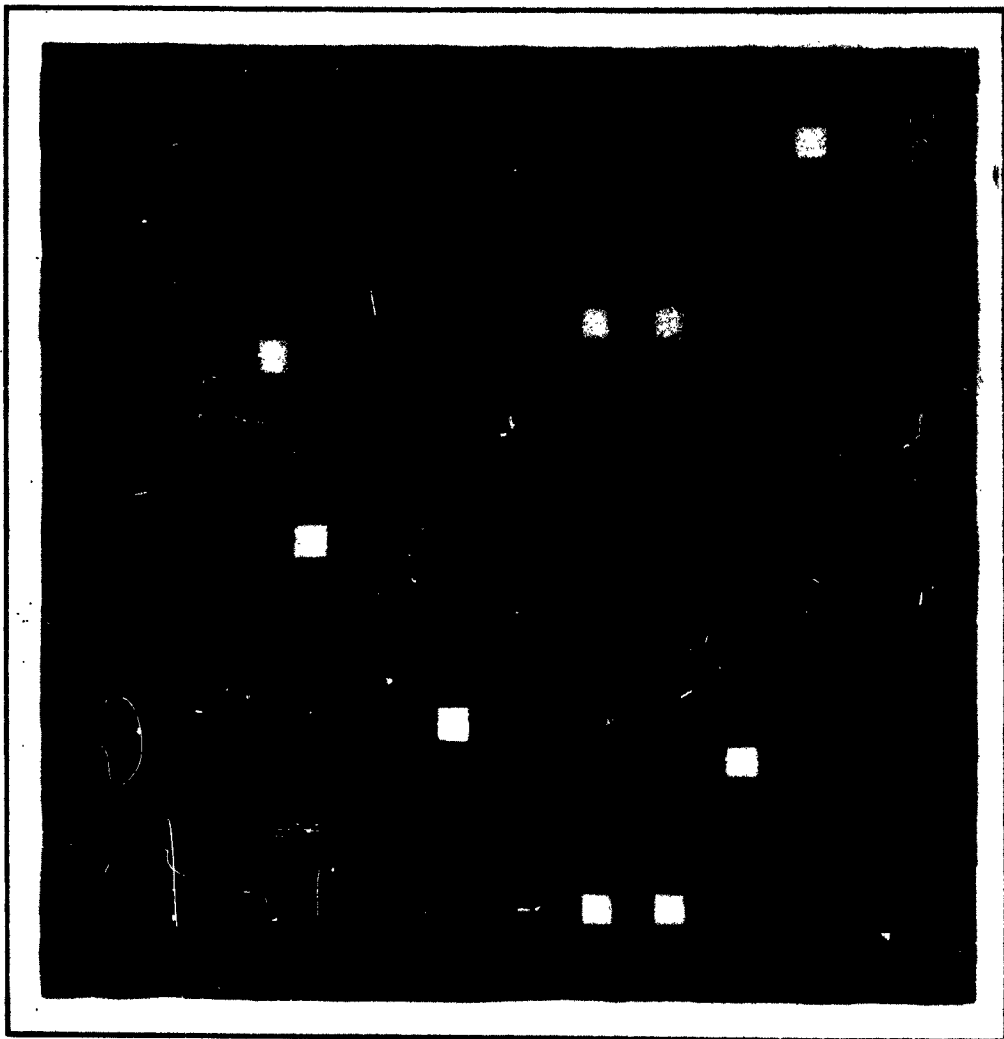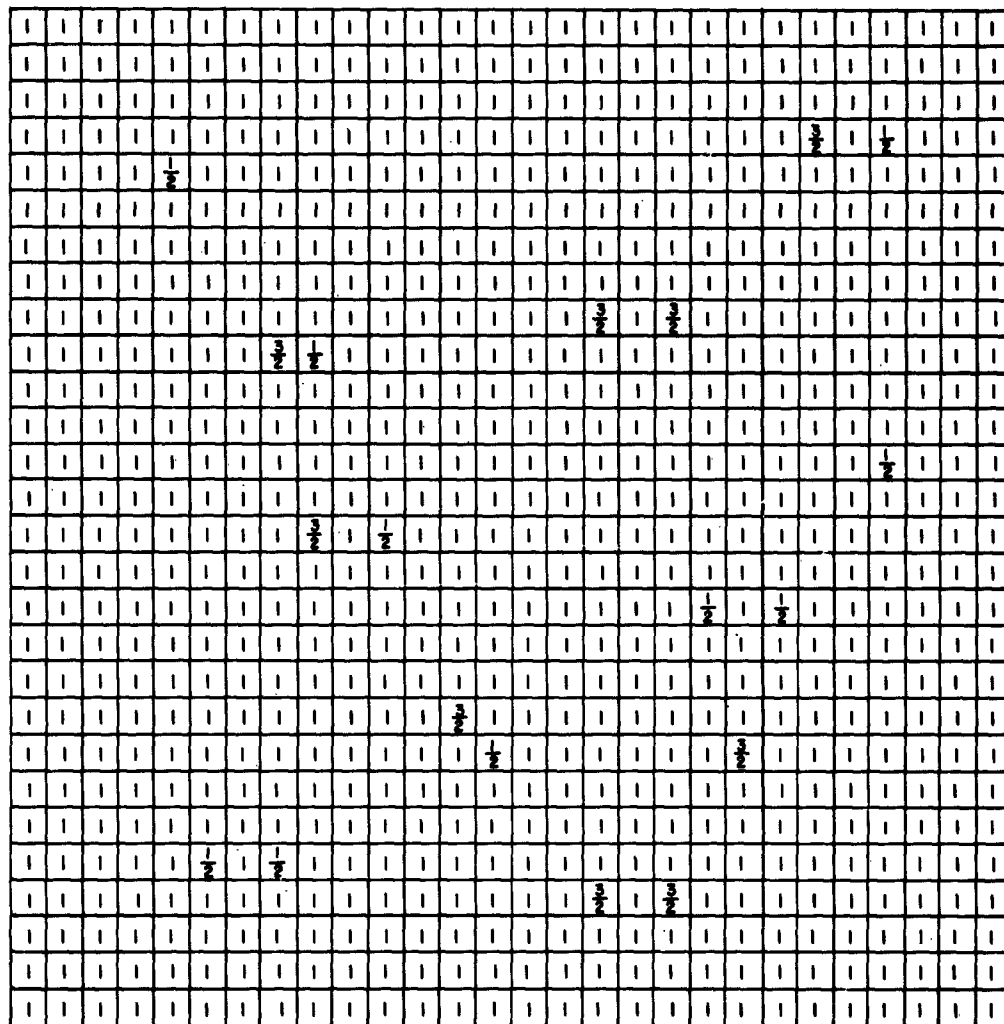Fig. 22. Y Matrix which results after filtering Fig. 21.

Fig. 23. Numerical intensity X Matrix for Fig. 21.

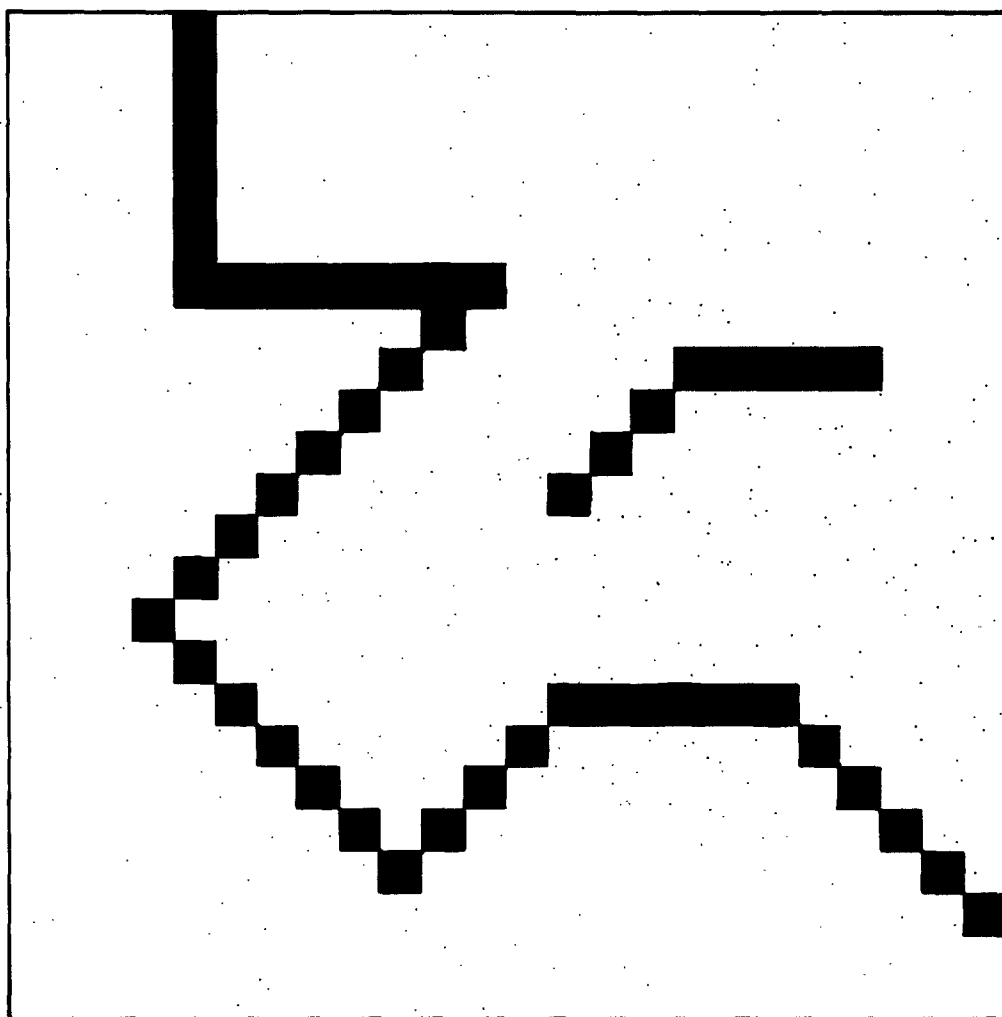Fig. 24. Numerical intensity Y Matrix for Fig. 22.

CHAPTER III
REALIZATION OF SPATIAL FILTERING

At the present time, all sensors which detect spatial-type information must scan the area of interest in some prescribed manner and the resulting information appears as a signal at a set of terminals with a single independent variable, namely time. If the original field varies as a function of time, each frame produced in the scanning process is essentially a single sample (samples in time) of the field. Because the area scanned usually has two spatial coordinates, it is necessary to establish a raster which defines the path followed by the sensor in scanning the area. This raster indicates the manner in which the area is sampled spatially. In spatial filtering, it will generally be necessary to store at least a section of the image. This can be accomplished dynamically by delay lines of various types, or by static means of storage such as magnetic cores, drums, or tapes.

A. Lens System

  1. Mechanical Scanning

By means of polarizing devices, the intensity of incident light can be varied. Consider the form that an A filtering matrix assumes - each cell has a discrete value. If an object (picture) has different light intensity levels corresponding to its features (for example, a television picture of a stationary scene), the light intensity from any

38

segment can be varied by polarization. As mentioned previously, a scanning process is required for spatial filtering.

A lens consisting of discrete segments, in which the orientation of the polarizing devices is varied to produce the relative values desired in an A matrix, could be used to scan an illuminated picture. A phototube attached to the lens would then sum the output from the filtered object and store this information in a suitable device (Y matrix).

A simplified form of this system is shown in Figure 25. The cross-hatched areas correspond to $\frac{1}{4}$, the horizontal-lined areas to $\frac{1}{2}$, and the clear area to 1 (no attenuation).

By scanning the entire picture and storing the output from each position on which the filter is centered, a spatially-filtered image can be realized.

This system has the disadvantage of being extremely slow as each frame must be mechanically scanned before another frame can be admitted to the filtering section.

2. Electronic Scanning

Another device that appears to have operating principles which could be adapted to spatial filtering is the charactron tube. This tube has a matrix through which the electron beam is directed to select desired characters for subsequent positioning on the face of the tube.
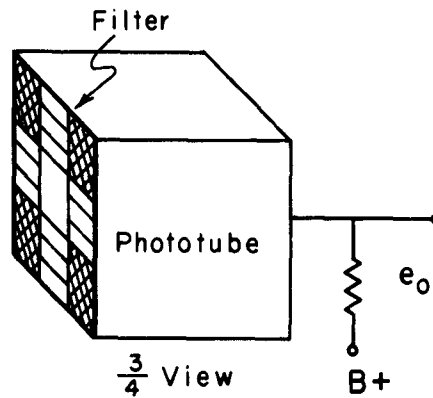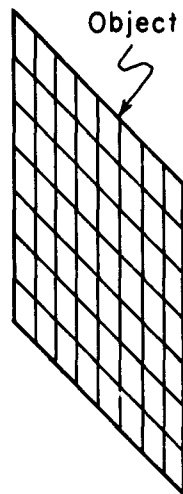
(a) Desired **A** Matrix
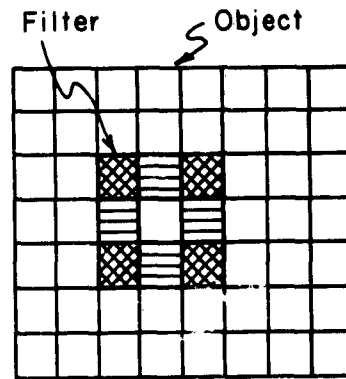
(b) Lens System In One Position For Filtering

(c) Output System

Fig. 25. Lens system for spatial filtering.

Now consider a modification to an image orthicon camera tube in which a matrix is positioned between the cathode and the target. This matrix could correspond to an A matrix by having each segment consist of a certain size mesh. Now, if an electron beam, which would completely cover the matrix, (consider the A matrix as a 3 x 3 matrix for this case) was directed through the matrix, the number of electrons going through each segment would be directly proportional to the mesh size. After passing through the matrix, the electron beam would then be deflected for the desired raster used in scanning the target. The center of the beam would correspond to the a_{oo} position of the A matrix and the point upon which it was directed on the target at any instant of time would correspond to the y_{ij} position of the Y matrix. The beam would thus cover 9 normal cells on the target. The number of electrons in the return beam would then be dependent upon the A matrix values and the charge distribution on the target. This would provide the desired multiplication of all cells adjacent to x_{ij} (the target is the X matrix) by the proper a_{pq} of the A matrix. The summing is done by the number of electrons in the return beam. This would provide a much more rapid means of scanning a target than the previous device consisting of a photo-tube and polarized lens.

41

CHAPTER IV
SPATIAL FILTERING WITH FEEDBACK

If a television camera is placed in front of its monitor so that the field of the camera exactly coincides with the size of the monitor screen, an x,y position of the monitor then corresponds to the same x,y position of the sensitized material in the camera. For example, a spot in the exact center of the monitor screen would be in the exact center of the image orthicon photo-cathode. Transmission by the camera would place the spot in the same original position on the monitor.

Now if the camera is displaced either to the right or left, a series of spots will be shown on the monitor with spacing between the spots equal to the camera displacement. This may be shown by a top view of the camera and screen:

(a) No Displacement   (b) Camera Displaced   (c) Camera Displaced
                          To The Right   To The Left
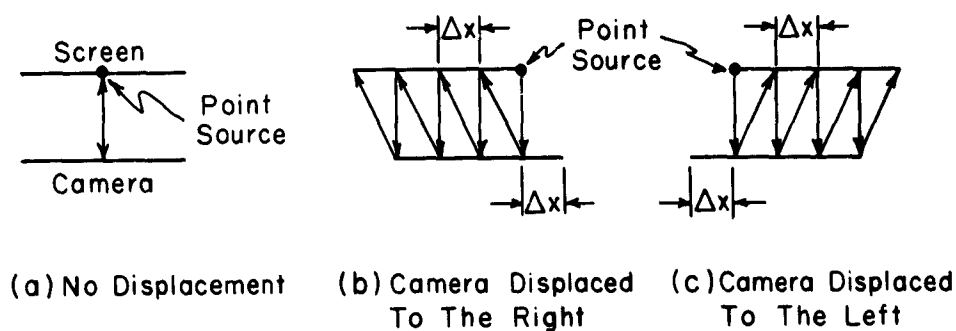
Fig. 26.   Top view of camera and monitor screen
           for spatial filtering with feedback.

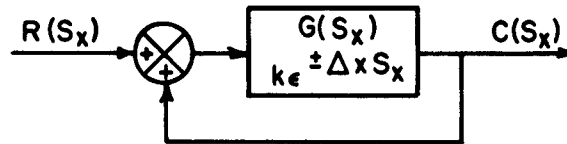This system is shown in Figure 27 as a servo diagram. Laplace transform for the transfer function is used.

Fig. 27. System servo diagram.

When the camera is displaced to the right a distance ΔX, the spots are displaced to the left by an amount ΔX. There is thus a shifting function $e^{+\Delta X S_x}$. For camera displacement to the left, the shifting function becomes $e^{-\Delta X S_x}$. The closed loop transfer function becomes

$$(54) \qquad \frac{C}{R} = \frac{k\, e^{\pm \Delta X S_x}}{1 - k e^{\pm \Delta X S_x}} \quad ; S_x = \alpha + j\beta$$

Two-sided Laplace transforms must be used for this system to define α for each direction. In time systems, instability would occur when the denominator equals zero. If such were the case here, for the positive exponent

$$e^{+\Delta X(\alpha + j\beta)} = \frac{1}{k}$$

$$(e^{+\alpha \Delta X})(e^{j\beta \Delta X}) = \frac{1}{k} \quad ; \quad \beta_n\, \Delta X = n 2\pi$$

$$\beta_n = \frac{n 2\pi}{\Delta X}$$

43

Since the shifting is to the left, ΔX can be considered negative and α_1 must therefore be positive for convergence of the Laplace integral . When shifting is to the right ΔX is positive and α_2 must be negative for convergence. Therefore, the region of convergence in the S_x plane is $\alpha_2 \leq 0 \leq \alpha_1$.

Actually instability is not a problem because the system is bounded. Intensity of the spots cannot increase beyond the percent modulation of the video signal in the camera for a pure white object.

CHAPTER V
CONCLUSIONS AND RECOMMENDATIONS

Spatial filtering is a process which may be utilized to highlight desired features in a field of view. A scanning process, which is a function of time, is required to process the field and store the information in suitable form. It has been shown in Reference 1 that a time varying signal may be stored in a matrix of transfluxors by signal steering techniques. Once this information is in matrix form, the spatial filtering process may be applied to eliminate redundancies, smooth fluctuations, or select desirable features from a background.

It has been shown that time filtering procedures using Laplace or Fourier transforms may be extended to include more than one variable, namely space coordinates. The restrictions of temporal filtering may be overcome with spatial filtering because the space coordinates may range over all values from plus infinity to minus infinity.

Spatial filters may be realized by means of lens systems, transfluxors, digital techniques, and other systems. Only two methods have been discussed here, but it is possible to devise such filters by any means which can multiply and sum the required elements. It is believed that the electronic scanning method could provide an extremely rapid means of accomplishing the desired filtering.

BIBLIOGRAPHY

1. Wolfe, Ramon E., "Signal Steering Techniques Employing Ferrite Cores", Report 820-1, August 1958, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract AF 33(616)-5590 with Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.

2. James, Hubert M., Nichols, Nathaniel B., and Phillips, Ralph S., Theory of Servomechanisms, McGraw-Hill Book Company, Inc., New York, 1947, pp. 232-245.

3. Van Der Pol, Balth. and Bremmer, H., Operational Calculus Based on the Two-Sided Laplace Integral, University Press, Cambridge, 1950.

4. Truxal, John G., Automatic Feedback Control System Synthesis, McGraw-Hill Book Company, Inc., New York, 1955, pp. 55-71, 500-518.

5. Gardner, Murray F. and Barnes, John L., Transients in Linear Systems, Volume I, John Wiley and Sons, Inc., New York, 1958, pp. 228-234, 262-263.

6. Robinson, David Z., "Methods of Background Description and Their Utility", Proceedings of the Institute of Radio Engineers, September 1959, pp. 1554-1561.

7. Aroyan, George F., "The Technique of Spatial Filtering",
    Proceedings of the Institute of Radio Engineers, September,
    1959, pp. 1561-1568.

8. Schwartz, Mischa, Information Transmission, Modulation, and
    Noise, McGraw-Hill Book Company, Inc., New York, 1959,
    pp. 18-79.

47